Calculus in Seashell Shape

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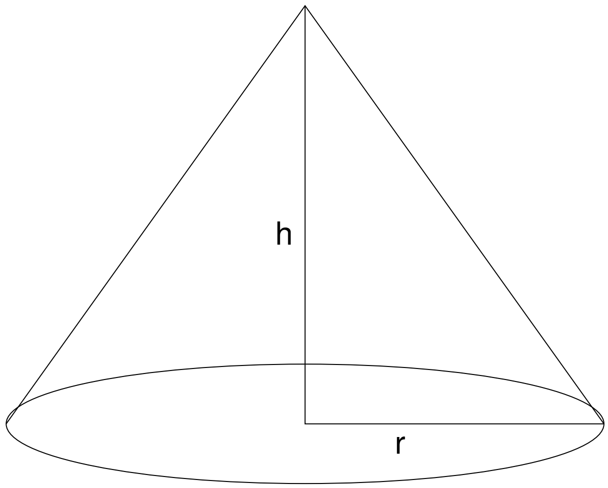
Here is a simple but beautiful example of calculus applied to a real-world biological problem, inspired by a recent talk by Tom Daniel. The problem has a very standard formulation from calculus, and should be an excellent homework problem for interested students. The approach presented here uses Lagrange multipliers and a slightly non-standard approach to the algebra, but this can be easily changed.

Consider yourself a mollusc living a half-billion years ago. You are a genius mollusc, and have discovered how to grow a conical shell on you back. There is one problem: you don't know what type of cone to make! You could make a long, narrow cone, but that may cramp your style. You could make a cone that looks like a cymbol, but then you'd spend all your time making shell and have no time to play. Or you could make something in the middle. What is the best shape to make your shell?

Ignoring other selective pressures(there probably is some reason other than style that you decided you needed a shell in the first place), you would like to minimizing its area and the amount of work you need to make it, while making sure you can still fit your body into it.



$V =$ your body volume.

$A =$ your conical shell's area.

$r =$ the radius of the base of your

$h =$ the height of your shell.

$s = h/r$, the ratio that controls the

$$V = \frac{1}{3}\pi r^2 h = \frac{\pi}{3} r^3 s \tag{1}$$

$$A = \pi r \sqrt{r^2 + h^2} = \pi r^2 \sqrt{1 + s^2} \tag{2}$$

We want to minimize the area A subject to the constraint that the volume V is keep constant. The Lagrange multiplier formulation of the problem is

$$L = \pi r^2 \sqrt{1 + s^2} + \lambda_1 \left(V - \frac{\pi}{3} r^3 s \right) \tag{3}$$

Any solution pair (r, h) must satisfy the necessary conditions

$$\frac{\partial L}{\partial s} = 0, \tag{4}$$

$$\frac{\partial L}{\partial r} = 0. \tag{5}$$

We get the following system of equations after differentiating:

$$\pi r^2 \frac{s}{\sqrt{1 + s^2}} - \lambda_1 \frac{\pi}{3} r^3 = 0 \quad , \tag{6}$$

$$2\pi r \sqrt{1 + s^2} - \lambda_1 \pi r^2 s = 0 \quad . \tag{7}$$

$$\frac{\pi}{3} r^2 \left(\frac{3s}{\sqrt{1 + s^2}} - \lambda_1 r \right) = 0 \quad , \tag{8}$$

$$\pi r \left(2\sqrt{1 + s^2} - \lambda_1 r s \right) = 0 \quad . \tag{9}$$

Eliminating $\lambda_1 r$ from the equations and assuming $r > 0$,

$$2\sqrt{1+s^2} - \frac{3s^2}{\sqrt{1+s^2}} = 0 \quad , \tag{10}$$

$$\frac{2(1+s^2) - 3s^2}{\sqrt{1+s^2}} = 0 \quad , \tag{11}$$

$$s = \sqrt{2}. \tag{12}$$

So, you want to choose $s = \sqrt{2}$ when you build your shell to maximize your volume-to-area ratio. This corresponds to an angle of about 109 degrees, 28 minutes, slightly bigger than a right cone.

While this may be the best solution when all you need to do is minimize effort, Nature is more complicated, and leads to much more complex shell shapes than just cones. Can people point out specific examples where the shells do appear to satisfy this criteria? What about cones formed by closed helices?